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SUBMANIFOLDS WITH RESTRICTIONS ON EXTRINSIC qTH SCALAR CURVATURE*

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ABSTRACT

We study the structure of the minimum set of the normal curvature for a symmetric bilinear map on Euclidean or Hilbert space, the conditions when this set contains strongly umbilical, conformal nullity, etc. linear subspaces. The main goals are estimates from above of the codimension of these subspaces for a symmetric bilinear map with positive normal curvature and the inequality type restriction on the extrinsic qth scalar curvature. We estimate from above the codimension of asymptotic and relative nullity subspaces for a symmetric bilinear map with nonpositive extrinsic qth scalar curvature.

Applying the algebraic results to the second fundamental form of a submanifold with low codimension, we characterize the totally umbilical and totally geodesic submanifolds, prove local nonembedding theorems for the products of Riemannian manifolds and global extremal theorem for the space of positive curvature. On the way we generalize results by Florit (1994), Borisenko (1977, 1987) and Okrut (1991) about Riemannian and Hilbert submanifolds.

Introduction

We first introduce some definitions and notations. Let M be a Riemannian space with the scalar product $\langle \cdot, \cdot \rangle$ on TM. The sectional curvature of M at a point $m \in M$ for a plane $\sigma \subset T_m M$ is denoted by $K_M(\sigma)$. Set

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$K_M(m) = \sup\{K_M(\sigma) : \sigma \subset T_mM\}$ and $K_M = \sup\{K_M(m) : m \in M\}$. In this paper all manifolds, maps, vector bundles, etc. are assumed to be of the class C^∞, unless otherwise stated.

Let $N \subset M$ be a submanifold with the second fundamental form

$$h\colon TN \times TN \to TN^\perp.$$

We call $|h(x,x)|$ the **normal curvature** of N in the direction $x \in TN$, $|x|$=1. The **mean curvature vector** of h is determined by the equality $H = \frac{1}{n}\,\mathrm{trace}\,h$. A submanifold N is said to be **totally geodesic** in M if $h = 0$, and **totally umbilical** if $h(x,y) = \langle x,y\rangle H$, where $x,y \in TN$ and $H \neq 0$. Define a continuous function $\lambda_h\colon N \to \mathbb{R}$ by

$$\lambda_h(m) = \min\{|h(x,x)|\colon x \in T_mN,\ |x| = 1\}.$$

We call $\lambda_h(N) = \inf_{m\in N} \lambda_h(m)$ the **minimum of the normal curvature** of N. The **minimum set of the normal curvature at** m is defined by

$$C_h(m) = \{x \in T_mN\colon |h(x,x)| = \lambda_h x^2\}.$$

Recall that a submanifold $N^n \subset M^{n+p}$ is **isotropic at** $m \in N$ if the normal curvature $|h(x,x)|$ is positive and does not depend on the choice of a unit vector $x \in T_mN$. In this case we have the equality $C_h(m) = T_mN$. Every totally umbilical submanifold is isotropic, but not vice versa.

In case of $\lambda_h(m) = 0$ the set $C_h(m)$ coincides with the set of all asymptotic vectors at m. In case of $\lambda_h(m) > 0$ there are no nonzero asymptotic vectors at m; moreover, if the inequality holds for all $m \in N$ then the function λ_h is C^∞-differentiable.

We say that $T_a \subset T_mN$ is an **asymptotic subspace** of N at m if $h(x,x) = 0$ for all $x \in T_a$. The dimension $\nu_a(m)$ of a maximal asymptotic subspace at m is called the **asymptotic index** of N at m. The **relative nullity subspace** $T_\nu(m) \subseteq T_mN$ at $m \in N$ is defined by the equality

$$T_\nu(m) = \{x \in T_mN\colon h(x,y) = 0,\ \forall y \in T_mN\}.$$

The dimension $\nu(m) = \dim T_\nu(m)$ is called the **relative nullity index** of N at m. The asymptotic index at m satisfies the inequality $\nu_a(m) \geq \nu(m)$. Set $\nu_a(N) = \min_{m\in N} \nu_a(m), \nu(N) = \min_{m\in N} \nu(m)$.

The structure of the set $C_h(m)$ in case of $\lambda_h(m) = 0$ (i.e., the cone of asymptotic vectors) was studied by several authors. The definition of the asymptotic index is due to A. Borisenko, the relative nullity index $\nu(N)$ was introduced by S. Chern and N. Kuiper. It is a well-known fact that the positiveness of $\nu(N)$ imposes strong conditions on the metric and on the structure of the submanifold. Borisenko studied the class of **strongly k-parabolic** submanifolds, i.e. with $\nu(N) \geq k$. He showed that the strongly k-parabolic submanifold N can be reconstructed in case when the base submanifold $B \subset N$ (i.e. transversal to the distribution T_ν) or the Grassmanian image of N have no asymptotic vectors.

S. Chern and N. Kuiper have shown that $\nu(m) \geq n-p$ at the points $m \in N$ where the extrinsic sectional curvature $K_h(\sigma) = K_N(\sigma) - K_M(\sigma)$ (the difference between the sectional curvatures of the submanifold and of the ambient space with respect to two-dimensional planes tangent to the submanifold) vanishes. For Riemannian submanifold $N \subset M$, the Gauss equation says that the **extrinsic sectional curvature** of N for a plane $\sigma \subset T_mN$ is given by

$$K_h(\sigma) = \langle h(x,x), h(y,y)\rangle - h^2(x,y), \tag{1}$$

where x, y is any orthonormal basis of σ (i.e., $\sigma = x \wedge y$). Set $K_h(m) = \sup\{K_h(\sigma) : \sigma \subset T_mN\}$. Otsuki [17] has proved that at each point m of a submanifold $N^n \subset M^{n+p}$ of codimension $p \leq n$ with $K_h(m) \leq 0$ there is at least 1-dimensional asymptotic subspace. Florit [9] generalized Otsuki's result by proving the inequalities

$$\text{a) } \nu_a(m) \geq n-p, \quad \text{b) } \nu(m) \geq n-2p. \tag{2}$$

These estimates are sharp. Let $U^2 \subset \mathbb{R}^3$ be a surface in Euclidean space with negative Gaussian curvature at $m_0 \in U^2$. Then the product immersion of p factors $N^n = U_1^2 \times \cdots \times U_p^2 \to \mathbb{R}^{3p}$ satisfies $\nu(m) = n-2p = 0$ at $m = (m_0, \ldots, m_0)$. Obviously, there is a p-dimensional asymptotic subspace $T_a(m) \subset T_mN$, i.e. the asymptotic index $\nu_a(m) = n-p$.

Definition 1: Let $V = \operatorname{span}\{x_1, \ldots, x_q\} \subseteq T_mN$ be a subspace spanned by q $(2 \leq q \leq n)$ orthonormal vectors at m. The **extrinsic qth scalar curvature of V** is defined by the formula

$$\tau_h^q(V) = \frac{2}{q(q-1)} \sum_{1\leq i<j\leq q} K_h(x_i \wedge x_j). \tag{3}$$

Set $\tau_h^q(N) = \sup_{m\in N} \tau_h^q(m)$, where $\tau_h^q(m) = \sup\{\tau_h^q(V) \colon V \subseteq T_mN,\ \dim V = q\}$.
The **q-dimensional extrinsic curvature** (q even) for V is defined by the formula, see [2],

$$\gamma_h^q(V) = \frac{1}{2^{q/2}q!}$$
$$\times \sum_{i,j\in S_q} \varepsilon(i)\varepsilon(j)[\langle h(x_{i_1},x_{j_1}),h(x_{i_2},x_{j_2})\rangle - \langle h(x_{i_1},x_{j_2}),h(x_{i_2},x_{j_1})\rangle]$$
$$\times\cdots\times[\langle h(x_{i_{q-1}},x_{j_{q-1}}),h(x_{i_q},x_{j_q})\rangle - \langle h(x_{i_{q-1}},x_{j_q}),h(x_{i_q},x_{j_{q-1}})\rangle]. \tag{4}$$

Here S_q stands for the set of all permutations of degree q and $\varepsilon(i)$ is the sign of a permutation $i = (i_1,\ldots,i_q)$. Set $\gamma_h^q(N) = \sup_{m\in N}\gamma_h^q(m)$, where $\gamma_h^q(m) = \sup\{\gamma_h^q(V)\colon V \subseteq T_mN,\ \dim V = q\}$.

These $\gamma_h^q(V)$ and $\tau_h^q(V)$ coincide with $K_h(V)$ for $q = 2$. The (intrinsic) qth scalar curvature $\tau_N^q(V)$ and the q-dimensional curvature $\gamma_N^q(V)$ are defined similarly to (3) and (4), using the sectional curvature K and the curvature tensor R of a manifold.

Recall that the **second fundamental tensor** of a normal $\xi \in TN^\perp$ is a linear self-adjoint operator $A_\xi\colon TN \to TN$ defined by the equality $\langle A_\xi x, y\rangle = \langle h(x,y),\xi\rangle$ $(x,y \in TN)$.

Borisenko [3] studied submanifolds $N^n\subset M^{n+p}$ with degenerate second fundamental tensor. He estimated the rank at $m \in N$ (i.e. the maximal rank of A_ξ for $\xi \in T_mN^\perp$) of a submanifold with $\gamma_h^q \le 0$ as

$$r(m) \le 2p(q-1). \tag{5}$$

For the relative nullity index Borisenko deduced a much stronger quadratic relation

$$\nu(m) \ge n - p(p+q-1). \tag{6}$$

However, his estimate (6) for $q = 2$ is weaker than (2b). Because of $r(m) \le n - \nu(m)$ the inequality (5) for $q = 2$ follows from (2b).

The paper deals with Riemannian submanifolds $N \subset M$ satisfying the inequality

$$\tau_h^q(m) \le \lambda_h^2(m) \tag{7}$$

for some q and $m \in N$. This condition generalizes the inequality $\tau_h^q(m) \le 0$.

Definition 2: Let $\lambda_h(m) > 0$. We say that $T_u(\xi) \subseteq C_h(m)$ is a **strongly umbilical subspace** relative to ξ, $|\xi| = \lambda_h(m)$ if

$$h(x,y) = \langle x,y\rangle\xi, \quad \forall x,y \in T_u(\xi).$$

The dimension $\nu_u(m)$ of a maximal strongly umbilical subspace at m is called the **strongly umbilical index** of N at m. We say that $\xi \in T_mN^\perp$, $|\xi| = \lambda_h(m)$, is the **strongly principal curvature normal** if the **strongly conformal nullity subspace** $T_c(\xi) \subseteq C_h(m)$ given by

$$T_c(\xi) = \{x \in T_mN\colon h(x,y) = \langle x,y\rangle\xi, \quad \forall y \in T_mN\}$$

is at least one-dimensional. An integer

$$\nu_c(m) = \min\{\dim T_c(\xi) : \xi \in T_mN^\perp, |\xi| = \lambda_h(m)\}$$

is called the **strongly conformal nullity index** of N at m. Set

$$\nu_u(N) = \min\{\nu_u(m) : m \in N, \lambda_h(m) > 0\},$$
$$\nu_c(N) = \min\{\nu_c(m) : m \in N, \lambda_h(m) > 0\}.$$

Remark 1: The following condition is sufficient for the inequality $\lambda_h(m) > 0$:

$$\forall x \in T_mN \setminus \{0\} \exists \sigma \ni x : \quad K_N(\sigma) > K_M(\sigma). \tag{8}$$

In fact, if $\exists x \in T_mN \setminus \{0\} : h(x,x) = 0$ (i.e. $\lambda_h(m) = 0$) then $K_N(\sigma) - K_M(\sigma) = K_h(\sigma) = -h^2(x \wedge y) \leq 0$ for all $\sigma \ni x$. For example, a submanifold $N \subset \mathbb{R}^N$ of positive sectional curvature at $m \in N$ obeys the condition $\lambda_h(m) > 0$.

We study the structure of the minimum set $C_h(m)$ in case of $\lambda_h(m) > 0$ and (7), the conditions when this set contains strongly umbilical and strongly conformal nullity (linear) subspaces. Our aim is to estimate from below the strongly umbilical and the strongly conformal nullity indices. The last index is a particular case of the conformal nullity index (and λ_h is a principal curvature function), see [18], that arises in several different geometric situations, see discussion in [7]. The strongly umbilical index $\nu_u(m)$ was defined for isometric immersions $N \subset M$ between space forms (in this case strongly umbilical subspaces were named isotropic subspaces), see [4]. Such immersions with **strong umbilical points**, i.e. $\nu_u(m) = \dim N$, are studied in [13], see also [16]. The equality $\nu_u(N) = \dim N$ characterizes totally umbilical submanifolds.

Our main goal is Theorem 1, which is based on a series of algebraic lemmas given in Section 1. Theorem 1 in case of $\lambda_h(m) > 0$ first estimates the indices $\nu_u(m)$ and $\nu_c(m)$, and in case of $\lambda_h(m) = 0$ improves (6) and generalizes the estimates (2). (For case of nonpositive extrinsic qth Ricci curvature see [19], [20]).

THEOREM 1: *Let $N^n \subset M^{n+p}$ be a Riemannian submanifold. Suppose that at $m \in N$ we have $\tau_h^q(m) \le \lambda_h^2(m)$ or $\gamma_h^q(m) \le \lambda_h^q(m)$ for some $q \le n/2$. Then:*

1. *If $\lambda_h(m) > 0$ and $p < \frac{n}{q-1}$, then*

$$\text{(9)} \qquad \text{a) } \nu_u(m) \ge n-(p-1)(q-1), \quad \text{b) } \nu_c(m) \ge n - 2(p-1)(q-1).$$

2. *If $\lambda_h(m) = 0$ and $p < \frac{n}{q-1} - 1$, then*

$$\text{(10)} \qquad \text{a) } \nu_a(m) \ge n - p(q-1), \quad \text{b) } \nu(m) \ge n - 2p(q-1).$$

Because of $r(m) \le n - \nu(m)$ inequality (5) follows from (10b). The cases (9b), (10b) of Theorem 1 hold if we omit the assumption for codimension p.

From the cases (9a), (10a) of Theorem 1 we conclude the following statement.

COROLLARY 1: *Let $N^n \subset M^{n+p}$ be a Riemannian submanifold. Suppose that at $m \in N$ we have $\tau_h^q(m) < \lambda_h^2(m)$ or $\gamma_h^q(m) < \lambda_h^q(m)$ for some $q \le n/2$. Then*

$$1)\ p \ge \frac{n-q}{q-1} \text{ if } \lambda_h(m) > 0; \quad 2)\ p \ge \frac{n-1}{q-1} \text{ if } \lambda_h(m) = 0.$$

Example 1: Let $N^n \subset M^{n+1}$ be a hypersurface with the unit normal field ξ and the second fundamental tensor A_ξ. Assume $\lambda_h(m) > 0$ at some $m \in N$. Then all eigenvalues $\{\lambda_i(m)\}$ of $A_\xi(m)$ have the same sign, since otherwise there is an asymptotic direction at m, a contradiction. Assume the eigenvalues are positive, $\lambda_i(m) \ge \lambda_h(m)$. Hence the matrix $A_\xi(m) - \lambda_h(m)E$ is nonnegative definite. From linear algebra follows $K_h(m) \ge \lambda_h^2(m)$. Hence $\tau_h^q(m) \ge \lambda_h^2(m)$ for each $q \ge 2$. The equality $\tau_h^q(m) = \lambda_h^2(m)$ means $A_\xi(m) = \lambda_h(m)E$, i.e. $\nu_u(m) = \nu_c(m) = n$. Hence our estimates (9) are sharp for $p = 1$.

A totally umbilical submanifold $N \subset M$ obeys the equality $K_h(m) = \lambda_h^2(m) > 0$ at each $m \in N$. An isotropic submanifold $N \subset M$ obeys the inequality $K_h(m) \le \lambda_h^2(m)$, and if equality holds at each point then N is totally umbilical, [15], [20]. From Theorem 1 (case 1) the corollary follows (see also [20] and Corollary 3 in Section 1.2).

COROLLARY 2: *Let $N^n \subset M^{n+p}$ be an isotropic submanifold. Then at each $m \in N$ we have*

$$\text{a) } nu_u(m) \ge n - (p-1), \quad \text{b) } nu_c(m) \ge n - 2(p-1).$$

An application of Theorem 1 (case 2) is Theorem 2 (local), see [9] for $q = 2$. We denote the Heaviside (unit step) function by $Heav(x) = \{\text{if } x < 0 \text{ then } 0, \text{ else } 1\}$.

Theorem 2: *Let $N^n = N_1 \times N_2$ be a product of Riemannian manifolds. Suppose that there exists $m = (m_1, m_2) \in N^n$ such that $K_{N_i}(\sigma) \geq 0$ ($\sigma \subset T_{m_i}N_i$) and $\tau^q_{N_i}(m_i) \leq 1$ ($i = 1, 2$) for some q. Then, there is no isometric immersion of N^n into a sphere $S^{n+p}(C)$ of constant sectional curvature $C = 1 + Heav(q-4)$ for $p < n/(2(q-1))$.*

This estimate for C is sharp, see Example 3. Basing on Theorem 1 (case 1) one may extend Theorem 2 to immersions without asymptotic vectors. The corresponding local Theorem 3 generalizes the result of [9], i.e. $\lambda_0 = 0$. For simplicity we assume $q = 2$ and drop the condition $K_{N_i}(\sigma) \geq 0$ ($\sigma \subset T_{m_i}N_i$).

Theorem 3: *Let $N^n = N_1 \times N_2$ be a product of Riemannian manifolds. Suppose that there exists $m = (m_1, m_2) \in N^n$ such that $K_{N_i}(m_i) \leq C + \lambda_0^2$ ($i = 1, 2$) for some $\lambda_0 > 0$ and $C > -\lambda_0^2$. Then, there is no isometric immersion of N^n into a space $M^{n+p}(C)$ of constant sectional curvature C for $\lambda_h(m) \geq \lambda_0$ and $p < n/2 + 1$.*

One may formulate Theorems 2,3 for the product $N^n = \prod_{i=1}^r N_i$ of $r > 2$ factors.

Further applications of Theorem 1, global Theorems 4–7, Corollary 4 for isotropic submanifolds and extremal Theorem 8, will be given in Sections 2.1–2.3. They characterize totally geodesic and totally umbilical submanifolds in Riemannian spaces of positive curvature by the inequality $\tau^q_h(N) \leq \lambda^2_h(N)$ or $\tau^q_h(N) \leq 0$. Similar applications to Hilbert submanifolds are given in Section 2.4.

1. Algebraic results on symmetric bilinear maps

Let $\mathbb{R}^N$ be an N-dimensional Euclidean space with a scalar product $\langle\cdot,\cdot\rangle$. Throughout Sections 1.1–1.3 we will denote by h: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$, $h(x,y) = h(y,x)$, a (vector-valued) symmetric bilinear map, where $n \geq 2$, $p \geq 1$. Section 1.4 deals with a continuous symmetric bilinear map h: $\mathbb{H} \times \mathbb{H} \to \mathbb{R}^p$ on a Hilbert space $\mathbb{H}$.

We will call $\lambda_h = \inf_{|x|=1} |h(x,x)|$ the **minimum of the normal curvature of** h, and denote the **minimum set of the normal curvature of** h by $C_h = \{x \in \mathbb{R}^n\colon |h(x,x)| = \lambda_h x^2\}$. If $\lambda_h > 0$ holds then h has no asymptotic vectors (i.e., nonzero elements of an asymptotic subspace), and if $\lambda_h = 0$ holds then C_h coincides with the cone of all asymptotic vectors of h. The **isotropic bilinear**

map h is defined by the equality $|h(x,x)| = \lambda_h x^2$, $\lambda_h \neq 0$. In this case $C_h = \mathbb{R}^n$ (or $C_h = \mathbb{H}$).

The **extrinsic sectional curvature of** h (with respect to two-dimensional plane σ of $\mathbb{R}^n$ or $\mathbb{H}$) is defined by the formula (1),

$$K_h(\sigma) = \langle h(x,x), h(y,y)\rangle - h^2(x,y),$$

where $\{x,y\}$ is any orthonormal basis of σ. Set $K_h = \sup_\sigma K_h(\sigma)$.

Let $V = \operatorname{span}\{x_1,\dots,x_q\}$ be a subspace spanned by q orthonormal vectors in $\mathbb{R}^n$ (or $\mathbb{H}$). The **extrinsic qth scalar curvature** $\tau_h^q(V)$ and the **extrinsic q-dimensional curvature** $\gamma_h^q(V)$ are defined by (3) and (4), resp. Set

$$\tau_h^q = \sup\{\tau_h^q(V)\colon \dim V = q\}; \quad \gamma_h^q = \sup\{\gamma_h^q(V)\colon \dim V = q\}.$$

Following the definition in Introduction we call a **subspace** $T_a \subseteq \mathbb{R}^n$ (or $T_a \subseteq \mathbb{H}$) **asymptotic** for h if $h(x,y)=0$ $(x, y\in T_a)$. We call $T_\nu(h) = \{x\colon h(x,y)=0,\ \forall y\}$ the **relative nullity subspace** of h.

Following Definition 2, we call a **subspace** $T_u(\xi) \subseteq C_h$ **strongly umbilical** relative to $\xi \in \mathbb{R}^p$ if $|\xi| = \lambda_h$ and $h(x,y) = \langle x,y\rangle\xi$ $(x,y \in T_u(\xi))$. We call a vector ξ of the length $|\xi| = \lambda_h$ the **strongly principal curvature normal** of h if the **strongly conformal nullity subspace** $T_c(\xi) = \{x\colon h(x,y) = \xi\langle x,y\rangle,\ \forall y\}$ is at least one-dimensional. From Proposition 1, given in what follows, we obtain that there exists at most one strongly principal curvature normal ξ of h with $\dim T_c(\xi) \geq 2$.

We study algebraic properties of a symmetric bilinear map h with the inequality $\tau_h^q \leq \lambda_h^2$ (or $\gamma_h^q \leq \lambda_h^q$). Note that each of these inequalities holds for the 2-nd fundamental form of isotropic submanifolds and becomes an equality for totally umbilical submanifolds. The results of Section 1 are applied in Section 2 and Introduction for the 2-nd fundamental form at a point of a submanifold. In Sections 1.1, 1.2 we estimate from below the strongly umbilical and strongly conformal nullity indices of h when p is small. Section 1.3 deals with the particular case of above inequality, nonnegative extrinsic qth scalar curvature (or nonnegative extrinsic q-dimensional curvature), we estimate from below the asymptotic and relative nullity indices of h when p is small relative to n. One may *replace the extrinsic qth scalar curvature by the extrinsic q-dimensional curvature* in all statements of Sections 1.1–1.4. For convenience, the results of Sections 1.1–1.3 are collected in Table 1.

$\tau_h^q \le 0$ or $\gamma_h^q \le 0$	$\tau_h^q \le \lambda_h^2$ or $\gamma_h^q \le \lambda_h^q$	Statement
$\nu_a \ge n - p(q-1)$	$\nu_u \ge n - (p-1)(q-1)$	Lemmas 2,5:a
$\nu \ge \nu_a - p(q-1)$	$\nu_c \ge \nu_u - (p-1)(q-1)$	Propositions 2,3
$\nu \ge n - 2p(q-1)$	$\nu_c \ge n - 2(p-1)(q-1)$	Lemmas 4,5:b

Table 1. Asymptotic and strongly umbilical indices

1.1 Strongly umbilical index of h. Let $h\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ be a symmetric bilinear map with the property $\lambda_h > 0$. The maximal dimension of strongly umbilical subspaces for h is denoted by $\nu_u(h)$ and is called the **strongly umbilical index of** h.

Proposition 1 (see [20]): *Let h: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ be a symmetric bilinear map with the property $\lambda_h > 0$. Suppose that for some orthogonal unit vectors x_1, x_2 from C_h we have $h(x_1, x_2) = 0$. Then $h(x_1, x_1) = h(x_2, x_2)$ and $K_h(x_1 \wedge x_2) = \lambda_h^2$.*

Proof of Proposition 1: . Assume the contrary $h(x_1, x_1) \ne h(x_2, x_2)$. The unit vector $x_0 = (x_1 + x_2)/\sqrt{2}$ is not parallel to x_1 or x_2. We have $h(x_0, x_0) = [h(x_1, x_1) + h(x_2, x_2)]/2$. Hence $|h(x_0, x_0)| < (|h(x_1, x_1)| + |h(x_2, x_2)|)/2 = \lambda_h$, a contradiction. ∎

Lemma 1: *Let h: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ be a symmetric bilinear map with the property $\lambda_h > 0$. Suppose that for some q we have the inequality $\tau_h^q \le \lambda_h^2$. If $p(q-1) < n$ holds then there is a strongly umbilical $(q-1)$-dimensional subspace X of h for some ξ; moreover, $K_h(x \wedge y) = \lambda_h^2$ for all $x \in X$ and $y \in \ker h(X, \cdot) = \{z \in \mathbb{R}^n\colon h(x, z) = 0\ \forall x \in X\}$.*

Proof of Lemma 1: The proof is based on the method used by Borisenko in the case $\gamma_h^q \le 0$.

The minimum λ_h^2 of a smooth positive function $f(x) = h^2(x, x)$ on the unit sphere S^{n-1} is reached on some unit vector x_1 that proves the lemma for $q = 2$. Now assume $q > 2$. Let $F(x) = f(x) - \mu\langle x, x\rangle$. From the necessary condition of an extremum for the vector x_1 we obtain

(11) $$\frac{1}{2} dF(x_1)x = 2\langle h(x_1, x_1), h(x_1, x)\rangle - \mu(x_1, x) = 0,$$

(12) $$\frac{1}{2} d^2F(x_1)(x, x) = 2\langle h(x_1, x_1), h(x, x)\rangle + 4h^2(x_1, x) - \mu\langle x, x\rangle \ge 0,$$

where $x \in \mathbb{R}^n$. From (11) we deduce that $(1/2)\mu = |h(x_1, x_1)|^2 = \lambda_h^2$ and that the subspace $V_1 = \ker h(x_1, \cdot)$ is orthogonal to x_1. Note that $\dim V_1 \geq n - p$ is positive. In view of (12), we have $\langle h(x_1, x_1), h(x, x)\rangle \geq (1/2)\mu$ for unit vectors $x \in V_1$.

The minimal value of $f(x)$ restricted on the unit sphere in V_1 is reached on some unit vector $x_2 \in V_1$. Let $V_2 = \ker h(x_2, \cdot) \subseteq V_1$ be the subspace of all vectors x such that $h(x_2, x) = 0$. Then $\dim V_2 \geq n - 2p$ is positive. As above, see (11) and (12), V_2 is orthogonal to x_2 and for the unit vectors $x \in V_2$ we have $\langle h(x_2, x_2), h(x, x)\rangle \geq |h(x_2, x_2)|^2 \geq |h(x_1, x_1)|^2$. Repeating the process $(q-2)$-times, we obtain a subspace V_{q-1} of positive dimension $\geq n - p(q-1)$. Let $x_q \in V_{q-1}$ be any unit vector. By construction, the unit vectors $\{x_1, \dots, x_q\}$ are mutually orthogonal, and we have

$$h(x_i, x_j) = 0, \quad \langle h(x_i, x_i), h(x_j, x_j)\rangle \geq |h(x_i, x_i)|^2, \quad 1 \leq i < j \leq q. \tag{13}$$

Set $V = \mathrm{span}\{x_1, \dots, x_q\}$. Hence

$$\lambda_h^2 \geq \tau_h^q(V) = \frac{2}{q(q-1)} \sum_{1 \leq i < j \leq q} \langle h(x_i, x_i), h(x_j, x_j)\rangle \geq |h(x_1, x_1)|^2 \geq \lambda_h^2.$$

From this it follows that $\langle h(x_i, x_i), h(x_j, x_j)\rangle = |h(x_i, x_i)|^2 = \lambda_h^2$ $(i < j \leq q)$. In view of Proposition 1, there is an $\xi \in \mathbb{R}^p$ with the property $|\xi| = \lambda_h$ such that $h(x_i, x_i) = \xi$ for $1 \leq i < q$. One can see that $X = \mathrm{span}\{x_1, \dots, x_{q-1}\}$ is a strongly umbilical $(q-1)$-dimensional subspace relative to ξ, and $h(X, V_{q-1}) = 0$. In fact, any unit vector $x \in X$ is presented in the form $x = \sum_{i=1}^{q-1} c_i x_i$, where $\sum_{i=1}^{q-1} c_i^2 = 1$. Hence $h(x, x) = \sum_{i=1}^{q-1} c_i^2 h(x_i, x_i) = \xi \sum_{i=1}^{q-1} c_i^2 = \xi$. ∎

Lemma 2: *Let $h\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ be a symmetric bilinear map with the property $\lambda_h > 0$. Suppose that for some q we have $\tau_h^q \leq \lambda_h^2$. If $p(q-1) < n$ then*

$$\nu_u(h) \geq n - (p-1)(q-1). \tag{14}$$

Lemma 2 completes Lemma 1 (see [9] for $K_h \leq 0$) and proves case 1(a) of Theorem 1.

Proof of Lemma 2: Define a linear transformation $h(x)\colon \mathbb{R}^n \to \mathbb{R}^p$ by $h(x)y = h(x, y)$ for $x \in C_h$. By Lemma 1 there is a strongly umbilical $(q-1)$-dimensional subspace $X_0 = \mathrm{span}\{x_1, \dots, x_{q-1}\}$ relative to some vector ξ_1 with the property $|\xi_1| = \lambda_h$. Also $K_h(\sigma) = \lambda_h^2$ for all planes $\sigma \subset X_0$. We set $V_1 = X_0 \oplus \tilde{V}_0$, $W_1 = \xi_1 \oplus \tilde{W}_0$ where

$$\tilde{V}_0 = \ker h(X_0, \cdot) = \bigcap_{1 \leq i < q} \ker h(x_i), \quad \tilde{W}_0 = \Big\{ \bigcup_{1 \leq i < q} \mathrm{Im} h(x_i) \Big\}^{\perp}.$$

Define $h_1 = h_{|V_1\times V_1}$. With the above notations we claim that $\mathrm{Im} h_1 \subseteq W_1$. To prove the claim, take unit vector $z \in \tilde V_0$ (i.e., $h(x_i, z) = 0$ for all i) and orthonormal vectors $y_i \in \mathbb{R}^n,\ y_i \perp X_0$, where $1 \le i < q$. From Lemma 1 follows that $\langle \xi_1, h(z,z)\rangle = K_h(x_i \wedge z) = \lambda_h^2$. Since $x_i \in C_h$ holds, we have for all t

$$\langle h(x_i + ty_i, x_i + ty_i), h(z,z)\rangle - h^2(x_i + ty_i, z) = \tag{15} \lambda_h^2 + 2t\langle h(x_i, y_i), h(z,z)\rangle + A_i t^2,$$

$$\langle h(x_i + ty_i, x_i + ty_i), h(x_j + ty_j, x_j + ty_j)\rangle - h^2(x_i + ty_i, x_j + ty_j) = \tag{16} \lambda_h^2 + 2t[\langle \xi_1, h(x_j, x_j)\rangle + \langle \xi_1, h(x_j, y_j)\rangle] + t^2 K_h(y_i \wedge y_j),$$

where A_i does not depend on t. We can assume $y_i \perp \ker h(x_i)$ (see the coefficient for t in (15), (16)). Thus the unit vectors $\tilde x_i = \frac{1}{\sqrt{1+t^2}}(x_i + ty_i)$ are orthogonal to z. Hence for $\tilde V = \{\tilde x_1, \dots, \tilde x_{q-1}, z\}$

$$\tau_h^q(\tilde V) = \lambda_h^2 + \frac{4t}{q(q-1)}\Big\langle \sum_{i=1}^{q-1} h(x_i, y_i),\ h(z,z) - (q-2)\xi_1 \Big\rangle + At^2, \tag{17}$$

where A does not depend on t. Note that the linear term in t in (17) changes sign under the transformation $x_i \to -x_i$ or $y_i \to -y_i$ for each i, but the equation still holds. In view of $\tau_h^q(\tilde V) \le \lambda_h^2$ this linear term vanishes, and we have for all i

$$\langle h(x_i, y),\ h(z,z) - (q-2)\xi_1\rangle = 0 \quad (y \in \mathbb{R}^n,\ y \perp X_0),$$

i.e., $h(z,z) \in W_1$ for all $z \in V_1$. In view of the symmetry of h, the claim is proved.

The above claim allows us to proceed inductively as follows. Set $V_0 = \mathbb{R}^n$, $W_0 = \mathbb{R}^p$ and $h_0 = h$. Given $k \ge 0$, for a symmetric bilinear map $h_k = h_{|V_k\times V_k}\colon V_k \times V_k \to W_k$ with the property $\tau^q_{h_k} \le \lambda_h^2 \le \lambda_{h_k}^2$, define a nonnegative integer (note that $\mathrm{Im} h_k(z)$ contains ξ_1)

$$r_k = \max\{\dim \mathrm{Im} h_k(z)\colon z \in A_{h_k}\} - 1,$$

and suppose that for $k \ge 1$

$$n_k = \dim V_k \ge n - (q-1)\sum_{i=0}^{k-1} r_i, \quad p_k = \dim W_k \le p - \sum_{i=0}^{k-1} r_i.$$

Note that $n_k - p_k(q-1) \ge n - p(q-1)$. Picking strongly umbilical (relative to $\xi_k,\ |\xi_k| = \lambda_{h_k}$) subspace $X_k = \mathrm{span}\{\hat x_1, \dots, \hat x_{q-1}\} \subset V_k$ such that

$$\max_{i<q} \dim \mathrm{Im} h_k(\hat x_i) = r_k + 1, \quad |h_k(\hat x_i, \hat x_i)| = \lambda_{h_k}, \quad h_k(\hat x_i, \hat x_j) = 0\ (i \ne j),$$

set $V_{k+1} = \bigcap_{1\le i<q} \ker h_k(\hat x_i)$, $W_{k+1} = \{\bigcup_{1\le i<q} \mathrm{Im} h_k(\hat x_i)\}^\perp$. Then

$$V_{k+1} \subset V_k \oplus X_k, \quad \dim V_{k+1} \ge n_k - (q-1)(r_k+1),$$
$$W_{k+1} \subset W_k \oplus \xi_k, \quad \dim W_{k+1} \le p_k - (r_k+1).$$

By Proposition 1, $\xi_k = h(\hat x_i, \hat x_i) = h(x_1, x_1) = \xi_1$. The above claim implies that $\mathrm{Im} h_{k+1} \subset W_{k+1}$, where $h_{k+1} = h_{|V_{k+1}\times V_{k+1}}$. Then

$$n_{k+1} = \dim V_{k+1} \ge (n_k + q - 1) - (q-1)(r_k+1) \ge n - (q-1)\sum_{i=0}^{k} r_i.$$

Because of

$$1 \le p_{k+1} = \dim W_{k+1} \le p_k + 1 - (r_k+1) \le p - \sum_{i=0}^{k} r_i,$$

there is an integer $m > 0$ such that $r_m = 0$. Thus the subspace $V_{m+1} = \bigcap_{1\le i<q} \ker h_m(\hat x_i)$ is strongly umbilical relative to ξ_m, i.e., the set $A_{h_m} = V_{m+1} \oplus X_m$ is a vector subspace. Note that $n_m - p_m(q-1) > 0$. By Lemma 1, each subspace $S \subseteq V_m$ with the dimension $\dim S > p_m(q-1)$ intersects A_{h_m} by a subspace of the dimension at least $q-1$. Hence, $\dim A_{h_m} \ge (n_m + q - 1) - p_m(q-1) \ge n - (p-1)(q-1)$. Founding on $h_m = h_{|V_m\times V_m}$, we conclude that A_{h_m} is a strongly umbilical subspace as required. ∎

Remark 2: In case of $\gamma_h^q \le \lambda_h^q$ we modify (15)–(17) as

$$\langle h(x_{i_1} + ty_{i_1}, x_{i_2} + ty_{i_2}), h(z,z)\rangle - \langle h(x_{i_1} + ty_{i_1}, z), h(x_{i_2} + ty_{i_2}, z)\rangle =$$
$$\lambda_h^2 \delta_{i_1}^{i_2} + t[\langle h(x_{i_1}, y_{i_2}), h(z,z)\rangle + \langle h(x_{i_2}, y_{i_1}), h(z,z)\rangle] + A_{i_1 i_2} t^2, \tag{18}$$

$$\langle h(x_{i_1} + ty_{i_1}, x_{j_1} + ty_{j_1}), h(x_{i_2} + ty_{i_2}, x_{j_2} + ty_{j_2})\rangle$$
$$-\langle h(x_{i_1} + ty_{i_1}, x_{j_2} + ty_{j_2}), h(x_{i_2} + ty_{i_2}, x_{j_1} + ty_{j_1})\rangle$$
$$= \lambda_h^2 (\delta_{i_1}^{j_1}\delta_{i_2}^{j_2} - \delta_{i_1}^{j_2}\delta_{i_2}^{j_1}) + 2tB + At^2, \tag{19}$$

where $A_{i_1 i_2}, A, B$ do not depend on t. We may assume $y_i \perp \ker h(x_i)$ (see the coefficient for t in (15), (16)). Then unit vectors $\tilde x_i = \frac{1}{\sqrt{1+t^2}}(x_i + ty_i)$ are orthogonal to z. Hence from (18), (19) we obtain

$$\gamma_h^q(\tilde V) = \lambda_h^q + \frac{4t\lambda_h^{q-2}}{q(q-1)}\Big\langle \sum_{i=1}^{q-1} h(x_i, y_i),\ h(z,z) - (q-2)\xi_1 \Big\rangle + At^2, \tag{20}$$

where $\tilde V = \{\tilde x_1, \ldots, \tilde x_{q-1}, z\}$, and A does not depend on t.

1.2 Strongly conformal nullity index of h. Let h: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ be a symmetric bilinear map with the property $\lambda_h > 0$. An integer $\nu_c(h) = \max_\xi \dim T_c(\xi)$ is called the **strongly conformal nullity index of** h. Following [13], we say that $y \in \mathbb{R}^n$ is a **regular element** of a bilinear map β: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$, if $\dim \mathrm{Im}\beta(y) = \max\{\dim \mathrm{Im}\beta(z)\colon z \in \mathbb{R}^n\}$. Note that the set $RE(\beta)$ of regular elements of β is open and dense in $\mathbb{R}^n$.

Lemma 3 ([13]): *Let β: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ be a bilinear map and $y_0 \in RE(\beta)$. Then $\beta(y, \ker(\beta(y_0)) \subseteq \mathrm{Im}\beta(y_0)$ for all $y \in \mathbb{R}^n$.*

We formulate the central result of the section that proves case 1(b) of Theorem 1.

Lemma 4: *Let h: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ be a symmetric bilinear map with the property $\lambda_h > 0$. Suppose that for some $q \le n/2$ we have $\tau_h^q \le \lambda_h^2$. Then*

$$\nu_c(h) \ge n - 2(p-1)(q-1). \tag{21}$$

Note that if $p > 1$, from $n - 2(p-1)(q-1) > 0$ follows $p(q-1) < n$. Lemma 4 (see [9] for $K_h \le 0$) is a corollary of Lemma 2 and the following proposition:

Proposition 2: *Let h: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ be a symmetric bilinear map with the property $\lambda_h > 0$. Suppose that for some $q \le n/2$ we have $\tau_h^q \le \lambda_h^2$, and for some ξ there exists a strongly umbilical subspace $T \subset T_u(\xi)$, $n - q \ge \dim T \ge q$. Then*

$$\nu_c(h) \ge \dim T - (p-1)(q-1).$$

Proof of Proposition 2: Note that from the conditions it follows that $q \le n/2$. Let $T' \subset \mathbb{R}^n$ be the orthogonal complement of T. Hence $\dim T' \ge q$. Define a bilinear map β: $T' \times T \to \mathbb{R}^p$ by $\beta = h_{|T' \times T}$. Take an orthonormal system of vectors $\{y_i\}_{1 \le i < q} \subset RE(\beta) \subset T'$ and let $x_0 \in W = \bigcap_{1 \le i < q} \ker \beta(y_i)$. Take an orthonormal system of vectors $\{x_i\}_{1 \le i < q} \subseteq T$, orthogonal to x_0. Let a unit vector $y \in T'$ be orthogonal to y_i for $1 \le i < q$. Using only the assumption that $K_h(x_i \wedge x_j) = \langle h(x_j, x_j), h(x_i, x_i)\rangle = \lambda_h^2$ and $h(x_j, x_i) = \delta_{ij}\xi$ on T and $h(x_0, y_i) = 0$ $(i > 0)$, we obtain for small $s, t \in \mathbb{R}$ that

$$\begin{aligned}&\langle h(x_0 + ty, x_0 + ty), h(x_i + sy_i, x_i + sy_i)\rangle - h^2(x_0 + ty, x_i + sy_i)\\ &\quad = \lambda_h^2 + 2s\{2t\langle h(x_0, y), h(x_i, y_i)\rangle + \langle h(x_i, y_i), \xi\rangle\} + 2t\langle h(x_0, y), \xi\rangle + A_i,\end{aligned} \tag{22}$$

$$\begin{aligned}&\langle h(x_j + sy_j, sy_j + x_j), h(x_i + sy_i, x_i + sy_i)\rangle - h^2(x_j + sy_j, x_i + sy_j)\\ &\quad = \lambda_h^2 + 2s\langle h(x_i, y_i) + h(x_j, y_j), \xi\rangle + A_{ij}, \quad i \ne j > 0,\end{aligned} \tag{23}$$

where A_i, A_{ij} contain the terms with s^2 or t^2. Define an orthonormal system of vectors $\tilde y_0 = \frac{1}{\sqrt{1+t^2}}(x_0+ty)$, $\tilde y_i = \frac{1}{\sqrt{1+s^2}}(x_i+sy_i)$ and set $\tilde V_{t,s} = \operatorname{span}\{\tilde y_i\}_{0\le i<q}$. Then:

$$\tau_h^q(\tilde V_{t,s}) = \lambda_h^2 + 2s[2t\langle h(x_0,y),\eta\rangle + \langle \xi,\eta\rangle] + 2t\langle h(x_0,y),(q-1)\xi\rangle + A, \tag{24}$$

where $\eta = \sum_{i=1}^{q-1} h(x_i,y_i)$ and A contains the terms with s^2 or t^2. Because of $\tau_h^q(\tilde V_{t,s}) \le \lambda_h^2$, one must equate to zero linear terms in s and t. This implies $\langle h(x_0,y),\xi\rangle = 0$, i.e.,

$$h(x_0,y)\perp\xi \Rightarrow \dim \operatorname{Im}\beta(y) \le p-1 \tag{25}$$

and

$$2t\langle h(x_0,y),\eta\rangle + \langle \xi,\eta\rangle = 0. \tag{26}$$

We equate to zero linear and free terms in t of (26) and obtain two equations

$$\langle h(x_0,y),\eta\rangle = 0, \quad \langle \xi,\eta\rangle = 0. \tag{27}$$

Note that each term $h(x_i,y_i)$ in η changes its sign under the transformation $x_i \to -x_i$ or $y_i \to -y_i$, but the equations (27) still hold. Thus from (27) we obtain

$$\langle h(x,y_i), h(x_0,y)\rangle = 0 \quad (x\perp x_0)$$

and, in view of $h(x_0,y_i)=0$, one can drop the assumptions $x\perp x_0$ and $y\perp\{y_i\}$. In view of (25) the dimension of the subspace $W = \bigcap_{1\le i<q} \ker\beta(y_i)$ obeys the inequality

$$\dim W \ge \dim T - (p-1)(q-1).$$

From the arbitrariness of $x\in T$ and $x_0\in W$ follows that

$$\beta(y,W)\perp \operatorname{Im}\beta(y_i) \quad (y\in T').$$

This, together with Lemma 3, tells us that $h(x,y)=0$ $(x\in W,\ y\in T')$. But since $W\subseteq T$ holds, we have $W\subseteq T_c(\xi)$ and then $\nu_c(\xi)\ge \dim W \ge \dim T-(p-1)(q-1)$. ∎

Remark 3: In case of $\gamma_h^q \le \lambda_h^q$ we modify (22)–(23) for $i_1,j_1,i_2,j_2>0$ to obtain

$$\begin{aligned}
&\langle h(x_0+ty,x_0+ty), h(x_{i_2}+sy_{i_2}, x_{j_2}+sy_{j_2})\rangle \\
&\quad -\langle h(x_0+ty,x_{i_2}+sy_{i_2}), h(x_0+ty,x_{j_2}+sy_{j_2})\rangle \\
&\qquad =\lambda_h^2\delta_{i_2}^{j_2} + s\{2t\langle h(x_0,y), h(x_{i_2},y_{j_2})+h(x_{j_2},y_{i_2})\rangle \\
&\qquad\quad + \langle h(x_{i_2},y_{j_2})+h(x_{j_2},y_{i_2}),\xi\rangle\} + 2t\delta_{i_2}^{j_2}\langle h(x_0,y),\xi\rangle + A_{i_2},
\end{aligned} \tag{28}$$

$$\begin{aligned}&\langle h(x_{i_1}+sy_{i_1},x_{j_1}+sy_{j_1}),h(x_{i_2}+sy_{i_2},x_{j_2}+sy_{j_2})\rangle\\ &\qquad -\langle h(x_{i_1}+sy_{i_1},x_{j_2}+sy_{j_2}),h(x_{i_2}+sy_{i_2},x_{j_1}+sy_{j_1})\rangle\\ &\qquad\qquad =\lambda_h^2(\delta_{i_1}^{j_1}\delta_{i_2}^{j_2}-\delta_{i_1}^{j_2}\delta_{i_2}^{j_1})+2sB+As^2,\end{aligned}\tag{29}$$

where A_{i_2} contains terms with s^2 or t^2, and A, B do not depend on s. We use the terms with st and t, the linear terms with s are not used. Define the vectors $\tilde y_0 = (x_0+ty)/\sqrt{1+t^2}$, $\tilde y_i = (x_i+sy_i)/\sqrt{1+t^2}$ and set $\tilde V = \operatorname{span}\{\tilde y_i\}_{0\le i<q}$. Then from (28), (29) we obtain

$$\gamma_h^q(\tilde V) = \lambda_h^q + \{s[2t\langle h(x_0,y),\eta\rangle + \langle \xi,\eta\rangle] + 2t\langle h(x_0,y),\alpha\xi\rangle\}\lambda_h^{q-2} + A,$$

where $\eta = \sum h(x_{i_2}, y_{j_2})$, $\alpha = const \in \mathbb{Z}$, and A contains the terms with s^2 or t^2.

Corollary 3: *Let $h\colon \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^p$ be an isotropic symmetric bilinear map. Then*

$$\nu_c(h)\ge n-2p+2.$$

Proof of Corollary 3: *In case of an isotropic symmetric bilinear map we have*

$$K_h(x\wedge y)\le\langle h(x,x),h(y,y)\rangle\le|h(x,x)|\cdot|h(y,y)|=\lambda_h^2$$

for all unit vectors $x\perp y$. The required inequality follows from Lemma 4 with $q=2$. ∎

1.3 Asymptotic and relative nullity indices of h.

Let

$$h\colon \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^p$$

be a symmetric bilinear map. Recall that if h has a nonzero asymptotic vector (i.e., element of an asymptotic subspace) then $\lambda_h = 0$. The **asymptotic index** $\nu_a(h)$ is defined as the maximal dimension of asymptotic subspaces. We call $\nu(h) = \dim T_\nu(h)$ the **relative nullity index** of h, where $T_\nu(h) = \{x\in\mathbb{R}^n\colon h(x,y)=0,\ \forall y\in\mathbb{R}^n\}$.

Basing on Lemmas 2 and 4 of Section 1.1, we estimate the asymptotic and relative nullity indices of h, and prove case 2 of Theorem 1.

Lemma 5: *Let $h\colon \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^p$ be a symmetric bilinear map. Suppose that for some $q\le n/2$ we have $\tau_h^q\le 0$. Then*

$$\nu(h)\ge n-2p(q-1).\tag{30}$$

If $(p+1)(q-1) < n$ *then*

$$\nu_a(h) \geq n - p(q-1). \tag{31}$$

Proof of Lemma 5: We first extend the $\mathbb{R}^p$-component of h to a symmetric bilinear map $\tilde h\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{p+1} = \mathbb{R}^p \times \mathbb{R}$ by the formula $\tilde h(x,y) = (h(x,y), \langle x, y\rangle)$. Because $|\tilde h(x,x)|^2 = |h(x,x)|^2 + 1$ for $|x| = 1$, we have $\lambda^2_{\tilde h} = \lambda^2_h + 1 \geq 1$. Note that $\tau^q_{\tilde h} = \tau^q_h + 1 \leq 1$ (or $\gamma^q_{\tilde h} = \gamma^q_h + 1 \leq 1$ in case of extrinsic q-dimensional curvature). Hence $\tau^q_{\tilde h} \leq \lambda^2_{\tilde h}$ (or $\gamma^q_{\tilde h} \leq \lambda^q_{\tilde h}$). Lemma 2 states that there is a strongly umbilical subspace $T \subseteq \mathbb{R}^n$ (for $\tilde h$ relative to some unit vector $\xi \in \mathbb{R}^{p+1}$) with normal curvature 1 and $\dim T \geq n - p(q-1)$. This implies that $\xi \perp \mathbb{R}^p$ and $h(x,x) = 0$ ($x \in T$). Hence T is an asymptotic subspace of h, and (31) is proven. The inequality $n - 2p(q-1) > 0$ yields $(p+1)(q-1) < n$. According to Lemma 4 and Proposition 3 we obtain (30). ∎

Proposition 3: *Let $h\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ be a symmetric bilinear map. Suppose that for some $q \leq n/2$ we have $\tau^q_h \leq 0$. Let $T \subseteq \mathbb{R}^n$, $n - q \leq \dim T \leq q$ be an asymptotic subspace of h. Then $\nu(h) \geq \dim T - p(q-1)$.*

Proposition 3 follows from Proposition 2, see method of proof in proof of Lemma 5.

1.4 Nullity of a symmetric bilinear map on a Hilbert space. Let $\mathbb{H}$ be a separable Hilbert space. In this section we study algebraic properties of a continuous vector-valued symmetric bilinear map $h\colon \mathbb{H} \times \mathbb{H} \to \mathbb{R}^p$, $h(x,y) = h(y,x)$, whose extrinsic qth scalar curvature and the normal curvature are related by the inequality $\tau^q_h \leq \lambda^2_h$ for some $q \geq 2$.

Proposition 4: *Let $h\colon \mathbb{H} \times \mathbb{H} \to \mathbb{R}^p$ be a continuous symmetric bilinear map with the property $\tau^q_h \leq \lambda^2_h$ for some $q \geq 2$. Then $C_h \neq 0$.*

Proof of Proposition 4: Assume the contrary; $|h(x,x)| > \lambda_h x^2$ for all $x \neq 0$. Let V be any $(p+2)$-dimensional subspace of $\mathbb{H}$. Set $\tilde h = h_{|V\times V}$, then $\lambda_{\tilde h} > \lambda_h \geq 0$. The minimum, $\lambda^2_{\tilde h}$, of a smooth positive function $f(x) = \tilde h^2(x,x)$ on the unit sphere $S^{n-1} \subset V$ is reached on some unit vector x_1. Let $F(x) = f(x) - \mu\langle x, x\rangle$. From the necessary condition of an extremum for the vector x_1 we obtain (11), (12), where $x \in V$. From (11) we deduce $(1/2)\mu = |\tilde h(x_1,x_1)|^2 = \lambda^2_{\tilde h}$ and that the subspace $V_1 = \ker \tilde h(x_1,\cdot)$ of V is orthogonal to x_1. Note that $\dim V_1 \geq n-p-1$ is positive. In view of (12), we have $\langle \tilde h(x_1,x_1), \tilde h(x,x)\rangle \geq (1/2)\mu$ for unit vectors $x \in V_1$. Hence $\lambda^2_h \geq K_h(x_1,x) = \langle \tilde h(x_1,x_1), \tilde h(x,x)\rangle \geq |\tilde h(x_1,x_1)|^2 = \lambda^2_{\tilde h} > \lambda^2_h$, a contradiction. ∎

Example 2: a) In Proposition 4, we cannot replace $\mathbb{R}^p$ by a Hilbert space $\mathbb{H}$. To show this, we construct a continuous symmetric bilinear map $h\colon \mathbb{H}\times\mathbb{H}\to\mathbb{H}$ (the 'normal space' is infinite dimensional) with $K_h\le 0$ and without asymptotic vectors. Let $\{e_i\}_{i\in\mathbb{N}}$ an orthonormal basis of $\mathbb{H}$. Starting from this base of the 'normal space' $\mathbb{H}$, one can build by induction a base $\{\xi_i\}_{i\in\mathbb{N}}$ of unit 'normals' with the property $\langle\xi_i,\xi_j\rangle<0$ for $i\ne j$. We may define a Hilbert symmetric bilinear form $h(x,y)=\sum_{i=1}^{\infty}x_iy_i\xi_i$, where $x=\sum_{i=1}^{\infty}x_ie_i$, $y=\sum_{i=1}^{\infty}y_ie_i$. Then $h(e_i,e_j)=\delta_{ij}\xi_i$ holds. Obviously, h has no nonzero asymptotic vectors. To prove that K_h is nonpositive, take two orthogonal unit vectors x and y and obtain

$$K_h(x\wedge y)=\left\langle\sum_{i=1}^{\infty}\xi_ix_i^2,\sum_{j=1}^{\infty}\xi_jy_j^2\right\rangle-\left(\sum_{i=1}^{\infty}\xi_ix_iy_i\right)^2=\sum_{i<j}\langle\xi_i,\xi_j\rangle(x_iy_j-x_jy_i)^2\le 0.$$

b) In Proposition 4, we cannot replace $\tau_h^q\le\lambda_h^2$ by weaker inequality $\tau_h^\infty\le\lambda_h^2$, where $\tau_h^\infty=\inf_{q\to\infty}\tau_h^q$. To show this assume that $\lambda_h=0$ and $p=1$. Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $\mathbb{H}$. One may define a scalar symmetric bilinear form $h(x,y)=\sum_{i=1}^{\infty}\frac{1}{2^i}x_iy_i$, where $x=\sum_{i=1}^{\infty}x_ie_i$, $y=\sum_{i=1}^{\infty}y_ie_i$. Then $0<\tau_h^q(V)\le\frac{2}{q(q-1)}\sum_{1\le i<j\le q}\frac{1}{2^{i+j}}$. Obviously that $\tau_h^\infty=0=\lambda_h$ but $h(x,x)>0$ for $x\ne 0$. Hence h has no asymptotic vectors.

Basing on Lemma 5, we show that asymptotic and relative nullity subspaces of $h\colon\mathbb{H}\times\mathbb{H}\to\mathbb{R}^p$ satisfying $\tau_h^q\le 0$ have the same codimension as for $q=2$.

LEMMA 6: *Let $h\colon\mathbb{H}\times\mathbb{H}\to\mathbb{R}^p$ be a symmetric bilinear map. If $\tau_h^q\le 0$ for some $q\ge 2$, then $K_h\le 0$. Moreover,*

(32) *a)* $\exists$ *asymptotic subspace* $T_a\subset\mathbb{H}:\operatorname{codim}T_a\le p$, *b)* $\operatorname{codim}T_\nu(h)\le 2p$.

Proof of Lemma 6: Take any two orthogonal unit vectors $x_1,x_2\perp T_\nu(h)$. Let $\tilde V$ be any $[2p(q-1)+q-2]$-dimensional subspace of $\mathbb{H}$ containing x_1,x_2. Set $\tilde h=h_{|\tilde V\times\tilde V}$. The nullity subspace $T_\nu(\tilde h)$ is orthogonal to x_1,x_2 and by Lemma 5, $\dim T_\nu(\tilde h)\ge q-2$. Take $q-2$ orthonormal vectors $\{e_i\}$ from $T_\nu(\tilde h)$. Denote $V=\operatorname{span}\{e_1,\dots e_{q-2},x_1,x_2\}$. Since $K_h(e_i\wedge\cdot)=K_{\tilde h}(e_i\wedge\cdot)=0$, we obtain $0\ge\tau_h^q(V)=\frac{2}{q(q-1)}K_h(x_1\wedge x_2)$. Hence $K_h(x_1\wedge x_2)\le 0$ as required.

To prove (a) assume the contrary: $\operatorname{codim}T_\nu(h)>2p$. Set $V_1=T_\nu^\perp(h)$; let $\tilde V$ be any $2p(q-1)$-dimensional subspace of $\mathbb{H}$ containing V_1. Set $\tilde h=h_{|\tilde V\times\tilde V}$, then $T_\nu(\tilde h)=T_\nu(h)\cap\tilde V$ and $V_1\perp T_\nu(\tilde h)$, $\dim V_1>2p$. By Lemma 5 for $q=2$, $\operatorname{codim}T_\nu(\tilde h)\le 2p$, a contradiction.

To prove (b) set $\tilde V=T_\nu^\perp(h)$ and $\tilde h=h_{|\tilde V\times\tilde V}$. Recall that $\dim\tilde V\le 2p$. By Lemma 5 for $q=2$, there is an asymptotic subspace $\tilde T_a$ of $\tilde h$ with $\operatorname{codim}\tilde T_a\le p$.

Then $T_a = \tilde{T}_a \oplus T_\nu(h)$ is an asymptotic subspace of h with $\operatorname{codim} T_a \le p$ as required. ∎

2. Applications to submanifolds

In this section we study the submanifolds $N \subset M$ satisfying the inequality $\tau_h^q(m) \le \lambda_h^2(m)$ (or $\gamma_h^q(m) \le \lambda_h^q(m)$), in particular, submanifolds with nonpositive extrinsic qth scalar (or q-dimensional) curvature. We characterize totally geodesic and totally umbilical submanifolds with small codimension, give corollaries for isotropic submanifolds, prove local nonembedding theorems for the products of manifolds and global extremal theorem about the sphere. One may replace the extrinsic qth scalar curvature by the extrinsic q-dimensional curvature (for even q) in all statements of Sections 2.1–2.3.

2.1 The qth scalar curvature of the product manifolds. In aim to prove Theorem 2 we need the following

Lemma 7: *Let N_1, N_2 be Riemannian manifolds. Suppose that there is a point $m = (m_1, m_2) \in N = N_1 \times N_2$ such that the conditions $K_{N_i}(\sigma) \ge 0$ $(\sigma \subset T_{m_i} N_i)$ and $\tau_{N_i}^q(m_i) \le 1$ (for some q) are satisfied. Then $\tau_N^q(m) \le 1 + Heav(q-4)$.*

One may generalize Lemma 7 to a product $N = \prod_{i=1}^r N_i$ of $r > 2$ factors. The estimate of τ_N^q in Lemma 7 is sharp for q even. This is obvious for $q = 2$, and for $q \ge 4$ consider the following example:

Example 3: Let N_i be the product $S^2(6) \times \mathbb{R}^2$ of a sphere and Euclidean plane. Then $\tau_{N_i}^4(m_i) < 2/(4(4-1)) \cdot 6 = 1$. Take the planes $V_1 = \operatorname{span}\{a_1, a_2\}$ in $T_{m_1}N_1$ and $V_2 = \operatorname{span}\{b_1, b_2\}$ in $T_{m_2}N_2$ tangent to S^2 factors and consider the subspace $V^4 = \operatorname{span}\{a_1, a_2, b_1, b_2\}$ of $T_m N$. Then the 4th scalar curvature $\tau_N^4(V) = 2/(4(4-1)) \cdot (6+6) = 2$. Analogous examples exist for any $q > 4$.

Proof of Lemma 7: Identify the tangent spaces $T_{m_i}N_i$ $(i = 1, 2)$ with corresponding subspace of the product $T_m N = T_{m_1}N_1 \times T_{m_2}N_2$. Let $T^q \subset T_m N$ be an arbitrary q-dimensional subspace at $m = (m_1, m_2)$. There exists an orthonormal basis $\{e_1, \dots e_q\}$ of T^q whose projections on the subspaces $T_{m_i}N_i$ are of the form $\cos\theta_{si}\vec{a}_{si}$ $(1 \le s \le q)$, i.e. $e_s = \cos\theta_{s1}\vec{a}_{s1} + \cos\theta_{s2}\vec{a}_{s2}$, where $\cos\theta_{s1}^2 + \cos\theta_{s2}^2 = 1$. Here $\{\vec{a}_{1i}, \dots, \vec{a}_{qi}\}$ are two orthonormal systems of vectors. Note that the vector $\vec{a}_{si} \in T_{m_i}N_i$ (for some s, i) is not uniquely determined if $\cos\theta_{si} = 0$. By curvature properties of the product manifolds, the sectional curvature $K_N(e_i \wedge e_j)$ of N is expressed in terms of sectional curvatures of the factors N_i as

(33) $$K_N(e_i \wedge e_j) = K_{N_1}(\vec{a}_{i1} \wedge \vec{a}_{j1}) \cos\theta_{i1}^2 \cos\theta_{j1}^2 + K_{N_2}(\vec{a}_{i2} \wedge \vec{a}_{j2}) \cos\theta_{i2}^2 \cos\theta_{j2}^2.$$

We denote the qth order symmetric square matrix whose diagonal is zero and other elements are nonnegative sectional curvatures $K_{s,ij} = K_{N_i}(\vec{a}_{is} \wedge \vec{a}_{js})$ by K_s. The assumption on curvature yields $\sum_{i<j} K_{s,ij} \le \frac{q(q-1)}{2}$ $(s = 1, 2)$. We introduce the nonnegative vectors $t_i = (\cos^2\theta_{1i}, \dots, \cos^2\theta_{qi})$ in the nonnegative octant $\mathbb{R}_+^q$ and note that

(34) $$t_1 + t_2 = \vec{1} = (1, \dots, 1) \in \mathbb{R}_+^q.$$

Then we present the qth scalar curvature $\tau_N^q(T^q) = 2/(q(q-1)) \sum_{i<j} K_N(e_i \wedge e_j)$ in the matrix form as the following quadratic function:

(35) $$\tau_N^q(T^q) = F(t_1, t_2) = \frac{2}{q(q-1)}(t_1 \cdot K_1 \cdot t_1^* + t_2 \cdot K_2 \cdot t_2^*).$$

The function $F(t_1, t_2)$ is convex in the product $\mathbb{R}_+^q \times \mathbb{R}_+^q$ of the nonnegative octants. Hence the restriction of F to the convex polyhedra

$$G = \{\mathbb{R}_+^q \times \mathbb{R}_+^q\} \cap \{t_1 + t_2 = \vec{1}\}$$

has its maximum at the vertex. The 2^q vertices of this polyhedra are presented by pairs of q-dimensional vectors consisting of ones and zeros. Associated to any vertex of G are two vectors: vector t_1 containing $\tilde{q}$ units and the complementary vector $t_2 = \vec{1} - t_1$ containing $q - \tilde{q}$ ones, say,

$$t_1 = (\underbrace{1, \dots 1}_{\tilde{q}}, 0, \dots, 0), \quad t_2 = (0, \dots, 0, \underbrace{1, \dots 1}_{q-\tilde{q}}).$$

Denote by $V_1 = \operatorname{span}\{e_1, \dots, e_{\tilde{q}}\} \subset T_{m_1}N_1$ and $V_2 = \operatorname{span}\{e_{\tilde{q}+1}, \dots, e_q\} \subset T_{m_2}N_2$ the subspaces corresponding to t_1 and t_2, and set $V^q = V_1 \oplus V_2$. The sums of the sectional curvatures along V_1 and V_2 obey the inequalities

(36)
$$S_1 = \sum_{1 \le i < j \le \tilde{q}} K_{N_1}(e_i \wedge e_j) \le \frac{q(q-1)}{2}, \quad S_2 = \sum_{\tilde{q} < i < j \le q} K_{N_2}(e_i \wedge e_j) \le \frac{q(q-1)}{2}.$$

Otherwise we will complete one of these subspaces, say V_1 when $S_1 > 1$, by vectors of V_1 to obtain q-dimensional subspace of $T_{m_1}N_1$ with qth scalar curvature larger than 1, a contradiction. In cases of $q = 2, 3$ either S_1 or S_2 vanishes. Thus the qth scalar curvature of the extremal subspace V^q obeys the inequality

$$\tau_N^q(V^q) = \frac{2}{q(q-1)}(S_1 + S_2) \le 1 + Heav(q-4). \qquad \blacksquare$$

Proof of Theorem 2: Case $q = 2$ was proven in [9] without the assumption $K_{N_i}(\sigma) \geq 0$ ($\sigma \subset T_{m_i}N_i$). Suppose that such immersion of N^n into a space form $M^{n+p}(C)$ exists for some $q > 2$. By Lemma 7, the product manifold $N = N_1 \times N_2$ satisfies the inequality $\tau_N^q(m) \leq C$ at $m = (m_1, m_2)$. By Theorem 1, case 2(b), $\nu(m) > 0$, i.e., there is a unit vector $y \in T_mN$ such that $h(y,z) \equiv 0$ and $K_N(y \wedge z) \equiv C$ for all unit vectors $z \in T_mN$ orthogonal to y. One may assume that the projection $\tilde{y}$ of y on the factor N_1 (say) is nonzero. Hence $K_N(\tilde{y} \wedge z) > 0$ for all unit vectors $z \in T_mN_2$. This is a contradiction since $K_N(\tilde{y} \wedge z) = 0$ for all $z \in T_mN_2$. ∎

Proof of Theorem 3: Suppose that such an immersion of N^n into a space form $M^{n+p}(C)$ exists. By Lemma 7 (for $q = 2$), the product manifold must satisfy $K_N(m) \leq C + \lambda_0^2$ at $m = (m_1, m_2)$. Hence $K_h(m) \leq \lambda_0^2 \leq \lambda_h^2(m)$. By Theorem 1, case 1(b), $\nu_c(\xi) > 0$ for some normal ξ at m with the property $|\xi| = \lambda_h(m)$. Thus there is a unit vector $y \in T_mN$ such that $h(y,z) \equiv \xi \cdot \langle y, z\rangle$ and $K_N(y \wedge z) \geq C + \lambda_0^2 > 0$ for all unit vectors $z \in T_mN$ orthogonal to y. One may assume that the projection $\tilde{y}$ of y on the factor, say N_1, is nonzero. Hence $K_N(\tilde{y} \wedge z) > 0$ for all unit vectors $z \in T_mN_2$. This is a contradiction since $K_N(\tilde{y} \wedge z) = 0$ for all $z \in T_mN_2$. ∎

Given a submanifold $N \subset \mathbb{R}^l$ with index $\nu(N) > 0$, it is interesting to ask *whether the relative nullity distribution gives rise to an Euclidean factor of the submanifold.*

We say that the scalar curvature τ_N of a Riemannian manifold N has **subquadratic growth along geodesics** if it satisfies $\lim_{t\to\infty} \tau_N(t)/t = 0$, where t is the parameter of any geodesic γ and $\tau_N(t)$ is the scalar curvature of N at $\gamma(t)$. The next result follows directly from Theorem 1 and Theorems 3, 4 of [6] (see [9] for $q = 2$).

THEOREM 4: *Let $N^{2n} \subset \mathbb{R}^{2n+p}$ be a minimal isometric immersion of a complete Kähler manifold of nonpositive qth scalar curvature for some $q \leq \frac{n}{2}$. Suppose that $2p(q-1) < n$ and one of the following holds:*

a) *the scalar curvature τ_N has subquadratic growth along geodesics, or*

b) *there exists $x_0 \in N^{2n}$ where all the holomorphic curvatures of planes in $T_\nu(x_0)^\perp$ are negative.*

Then $N^{2n} = N_1^{2p(q-1)} \times \mathbb{R}^{2n-2p(q-1)}$ and $f = f_1 \times Id$ splits.

2.2 Tests for totally umbilical submanifolds. **Curvature-invariant** submanifolds $N \subset M$ are defined by the condition

$$R_M(x,y)z^{\perp} = 0 \quad (x,y,z \in TN) \tag{37}$$

(which is obviously satisfied for the submanifolds in space forms).

The following is a special case of the result in [18].

Proposition 5: *Let $N \subset M$ be a curvature-invariant submanifold with $\lambda_h(N) > 0$. Suppose that ξ is a continuous strongly principal curvature normal (vector field) with the property $\nu_c(\xi(m)) \geq s$ for all $m \in N$. Let $G_\xi \subset N$ be an open set of N on which the codimension of the strongly conformal nullity subspace $T_c(\xi)$ relative to ξ is maximal. Then the following hold:*

(1) *$T_c(\xi)$ is an integrable strongly umbilical distribution on G_ξ, whose leaves are (pieces of) s-dimensional totally umbilical submanifolds in M if and only if ξ is parallel in the normal connection of $TN^{\perp}$ along $T_c(\xi)$.*

(2) *If $s > 1$ then ξ is parallel in the normal connection of $TN^{\perp}$ along $T_c(\xi)$ on G_ξ.*

(3) *The leaves are totally geodesic in G_ξ if and only if $\nabla_x(\lambda_h) = 0$ for $x \perp T_c(\xi)$.*

Moreover, if M is complete then the leaves (on a domain G_ξ) are complete.

From Theorem 1 (case 2) and Proposition 5 we deduce Theorem 5, which completes results in [3]. The assumption on the sectional curvature in Theorem 5 can be replaced by the weaker one for sth Ricci curvature, see [19], [20].

Theorem 5: *Let $N^n \subset M^{n+p}$ be a complete curvature-invariant submanifold. Suppose that $\lambda_h(N) > 0$ and $\tau_h^q(N) \leq \lambda_h^2(N)$ are valid for some $q \leq n/2$. If one of the conditions following holds,*

a) *$K_N(\sigma) > 0$ ($\sigma \subset TN$) and $p < (n+1)/(4(q-1)) + 1$,*

b) *$K_M(\sigma) > 0$ ($\sigma \subset TM_{|N}$) and $p < n/(4q-3) + 1$,*

then N^n is a totally umbilical submanifold.

Proof of Theorem 5: From the conditions it follows that $p(q-1) < n$. Assume that N is not a totally umbilical submanifold, i.e., $\nu_c(\xi) < n$. By Theorem 1, $\nu_c(\xi) \geq n - 2(p-1)(q-1)$ for some continuous normal vector field ξ of length $\lambda_h(N)$. The leaves $\{L\}$ of the strongly conformal nullity distribution are totally umbilical submanifolds in M of normal curvature $\lambda_h(N)$ (see Proposition 5), and are totally geodesic in N. Let L_1, L_2 be two sufficiently close leaves. The shortest geodesic $\gamma(t)$ $(0 \leq t \leq 1)$ with length $l = \mathrm{dist}(L_1, L_2)$ between the points $m_i \in L_i$ is orthogonal to L_1 and L_2. Since normals $\xi_1 = \dot\gamma\,(0), \xi_2 = \dot\gamma\,(1)$

to L_1, L_2 are tangent to N and orthogonal to mean curvature vectors of leaves, we have $h_i(x_i, x_i) \perp \xi_i$, where $x_i \in T_{m_i} L_i$. We have $2\nu_c(\xi) > n - 1$ for a), and $2\nu_c(\xi) > n + p - 1$ for b). Hence, in both cases, a) and b), there is unit vector $y_1 \in T_{m_1} L$ such that its parallel translation $y_1(t)$ along γ belongs to $T_{m_2} L$.

The formula for 2^{nd} variation of the energy E of γ along the parallel vector field $y_1(t)$, in N for a) and in M for b), is reduced to

$$a)\ \frac{1}{2}E''(y_1) = -\int_0^1 K_N(\dot\gamma \wedge y_1(t))dt, \quad b)\ \frac{1}{2}E''(y_1) = -\int_0^1 K_M(\dot\gamma \wedge y_1(t))dt.$$

From $E''(y_1) \geq 0$ we get a contradiction to positiveness of curvature K_N or K_M. ∎

The isotropic submanifolds with low codimension have not been studied much, see [15] and [5] in the case of isometric immersions between space forms. An isotropic submanifold $N \subset M$ is called a **constant isotropic submanifold** if the function λ_h: $N \to \mathbb{R}$, see Section 2.2, is constant. From Theorem 5 (with $q = 2$) we conclude the following statement about constant isotropic submanifolds.

COROLLARY 4 (see [20]): *Let $N^n \subset M^{n+p}$ be a complete curvature-invariant constant isotropic submanifold. If one of the following conditions holds:*
a) $K_N(\sigma) > 0$ $(\sigma \subset TN)$ *and* $p < \frac{n+5}{4}$,
b) $K_M(\sigma) > 0$ $(\sigma \subset TM_{|N})$ *and* $p < \frac{n+5}{5}$,
then N is a totally umbilical submanifold.

Note that isotropic isometric immersions between space forms $N^n(c) \subset M^{n+p}(\tilde c)$ satisfying the conditions $c \geq \tilde c$ and $p < \frac{n(n+1)}{2}$ are totally umbilical [5].

2.3 SUBMANIFOLDS WITH NONPOSITIVE EXTRINSIC qTH SCALAR CURVATURE. From case 2 of Theorem 1 following the proof of Theorem 5 we conclude:

THEOREM 6: *Let $N^n \subset M^{n+p}$ be a complete curvature-invariant submanifold. Suppose that for some $q \leq n/2$ we have $\tau^q_h(N) \leq 0$ or $\gamma^q_h(N) \leq 0$. If one of the following conditions holds:*
a) $K_N(\sigma) > 0$ $(\sigma \subset TN)$ *and* $p < \frac{n+1}{4(q-1)}$,
b) $K_M(\sigma) > 0$ $(\sigma \subset TM_{|N})$ *and* $p < \frac{n+1}{2q-1}$,
then N is a totally geodesic submanifold.

By a result of D. Ferus [8], *any complete Riemannian submanifold N^n of a round sphere S^N with the property $\nu(N) > F(n)$ must be totally geodesic.*

The **Ferus numbers** $F(n)$ are defined as $F(n) = \max\{t\colon t < \rho(n-t)\}$. Here $\rho((\mathrm{odd})2^{4b+c}) = 8b + 2^c$ $(b \geq 0,\ 0 \leq c \leq 3)$, i.e., $\rho(N) - 1$ is the **maximal number of continuous pointwise linearly independent vector fields on the sphere** S^{n-1}. In particular, $F(n) \leq 8d - 1$ for $n < 16^d$, and $F(2^d) = 0$. *Any complete Kähler submanifold N of CP^N with $\nu(N) > 0$ is totally geodesic*, [1].

Basing on the above facts and Theorem 1, we generalize Florit's result.

THEOREM 7: *Let $N^n \subset M^{n+p}$ be a complete Riemannian submanifold, and the curvature tensor obeys*

$$R_M(x,y)x = -kyx^2 \quad (x, y \in TN,\ k = const > 0). \tag{38}$$

Suppose that for some $q \leq n/2$ we have $\tau_h^q(N) \leq 0$ or $\gamma_h^q(N) \leq 0$. If one of the following conditions holds:

$$a)\ p < \frac{n - F(n)}{2(q-1)}, \quad b) N, M \text{ are Kähler and } p < \frac{n}{2(q-1)},$$

then N is a totally geodesic submanifold.

Note that (38) is stronger than (37), and it is obviously satisfied for submanifolds in a sphere of curvature k. In case of stronger restrictions on the curvature of M in Theorem 6 we obtain the global extremal theorem:

THEOREM 8: *Let M^{n+p} be a simply connected Riemannian space with the sectional curvature $1 \leq K_M(\sigma) \leq 9/4$ $(\sigma \subset TM)$, and N^n be a compact curvature-invariant submanifold. If qth scalar curvature (or q-dimensional curvature) of N is less than or equal to 1 for some $q \leq n/2$, and $p \leq (n-q)/(2(q-1))$, then M^{n+p} is isometric to the unit sphere.*

The assertion fails if the condition on the curvature is not satisfied. For instance, the complex projective space with curvature $1 \leq K_M(\sigma) \leq 9/4$ $(\sigma \subset TM)$ contains totally geodesic submanifolds that are globally isometric to a real projective space of constant curvature 1. Theorems 6–8 are generalizations of results in [3], where $q = 2$, $\gamma_h^q(N) \leq 0$ and a much stronger restriction on the curvature tensor is assumed.

Proof of Theorem 8: Let $G_\nu \subseteq N$ be an open domain where the relative nullity index is minimal. It is well-known (see, e.g., [12]) that the minimum relative nullity distribution T_ν is smooth and integrable on G_ν, and the leaves $\{L\}$ are $\nu(N)$-dimensional totally geodesic submanifolds in both N and M. If, in addition, N is complete, then the leaves are also complete. Because of $\tau^q(V) \leq 1$ and $\tau^q_M(V) \geq 1$ for $V \subset TM_{|N}$, we have $\tau^q_h(N) \leq 0$. Hence from Theorem 1 (case 2b) follows that $\nu(N) \geq q$. The scalar curvature of any q-dimensional subspace $V \subseteq T_\nu(m)$ satisfies the equalities $\tau^q_N(V) = 1 = \tau^q_M(V)$. Hence the relative nullity foliation $\{L\}$ $(TL = T_\nu)$ on a domain G_ν satisfies the equalities

$$K_M(x \wedge y) = K_N(x \wedge y) = 1 \quad (x, y \in T_\nu).$$

Thus, M contains a $\nu(N)$-dimensional totally geodesic submanifold L_0 of curvature 1. The rest of the proof is similar to [3] for $q = 2$. We need the following

LEMMA 8 ([3]): *Let M be a compact simply connected Riemannian C^4-manifold with the sectional curvature $1 < K_M(\sigma) \leq 4$ $(\sigma \subset TM)$. Then a complete totally geodesic submanifold L^ν $(\nu \geq 2)$ is simply connected.*

By Lemma 8 and the curvature restrictions on M, a submanifold L_0 is simply connected. Hence it is isometric to the unit $\nu(N)$-dimensional sphere. Thus M contains a closed geodesic with length 2π. If M is a simply connected Riemannian manifold with the curvature $0 < 1/4c < K_M(\sigma) \leq c$ for all $\sigma \subset TM$ then the injectivity radius $r_{in}(M) \geq \pi/\sqrt{c}$, [10], and every geodesic with the length $\leq r_{in}(M)$ is the shortest one. In our case of $c = 9/4$, every geodesic with the length $2/3\pi$ is the shortest one. Let m_1, m_2, m_3 be the points on this geodesic that define the triangle with equal sides $2/3\pi$. Then we apply Toponogov's theorem (see below) to this triangle with vertices m_1, m_2, m_3 and obtain that M is isometric to the unit sphere. ∎

THEOREM 9 ([21]): *Let a compact Riemannian C^4-manifold M with the sectional curvature $K_M(\sigma) \geq 1$ $(\sigma \subset TM)$ contains a triangle of perimeter 2π whose sides are the shortest geodesics. Then M is isometric to the unit sphere.*

2.4 APPLICATIONS TO HILBERT SUBMANIFOLDS. In this section we apply the algebraic results of Section 1.4 to the 2-nd fundamental form of a Hilbert submanifold with the first normal space of finite dimension. All manifolds and fiber bundles under discussion are Hilbert of the class C^∞, model or a tangent space (fibre) is a separable Hilbert space (i.e. l_2) denoted by $\mathbb{H}$, and maps between them are smooth morphisms. Up to an isometry, there is only one complete simply connected Hilbert manifold with given constant positive or negative sectional curvature modeled on a given Hilbert space. In contrast, one has to give an additional structure, in order to define scalar (or Ricci) curvature of an infinite dimensional Hilbert manifold, see [11].

The usual model of a **Hilbert sphere** is a totally umbilical hypersurface $S^\infty(C) = \{x \in \mathbb{H} : \langle x, x\rangle = 1/C\}$ of $\mathbb{H}$ with the induced Riemannian metric. It is a complete manifold of a constant sectional curvature C. A **small sphere** $N \simeq S^\infty(c)$ (of curvature $c > C$) in a Hilbert sphere $S^\infty(C)$ with the codimension $p < \infty$ can be presented by the equations

$$x_1 = \sqrt{1/C - 1/c}, \quad x_2 = \cdots = x_p = 0, \quad \sum_{i=1}^{\infty} x_{p+i}^2 = 1/c.$$

Let $N \subset M^\infty$ be a submanifold with the second fundamental form $h\colon TN \times TN \to TN^\perp$. The **first normal space** T_mN^1 of $T_mN^\perp$ is spanned by the image of h at m, that is,

$$T_mN^1 = \operatorname{span}\{h(x,y) : x, y \in T_mN\} \subset T_mN^\perp.$$

Hilbert submanifolds of **finite codimension**, $\dim T_mN^\perp < \infty$, (**hypersurfaces** when $\dim T_mN^\perp = 1$) serve as example of submanifolds with $\dim T_mN^1 < \infty$ for all $m \in N$.

From Lemma 6 we conclude the following statement analogous to Theorem 1.

Theorem 10: *Let $N \subset M^\infty$ be a Hilbert submanifold with $\dim T_mN^1 = p < \infty$.*

1. *If $\lambda_h(m) > 0$ and $\tau_h^q(m) \le \lambda_h^2$ (or $\gamma_h^q(m) \le \lambda_h^q$) for some $q \ge 2$ and $m \in N$, then there exists ξ such that*

$$\text{(39)} \qquad a)\ \operatorname{codim} T_u(\xi) \le (p-1)(q-1), \quad b)\ \operatorname{codim} T_c(\xi) \le 2(p-1)(q-1).$$

2. *If $\tau_h^q(m) \le 0$ for some $q \ge 2$ and $m \in N$, then $K_h(m) \le 0$ and*

$$\text{(40)} \qquad a)\ \operatorname{codim} T_a(m) \le p, \quad b)\ \operatorname{codim} T_\nu(m) \le 2p.$$

Okrut [14] showed that a curvature-invariant submanifold in a Hilbert manifold M^∞ with finite codimension and $K_h \le 0$ has at each point nonzero relative nullity subspace. He applied this to a complete submanifold in a Hilbert sphere S^∞ proving that such a submanifold is a large sphere.

Using Theorem 10 and Proposition 5 (which may be adopted for Hilbert submanifolds) we conclude the following statement that generalizes result in [14].

Theorem 11: *Let $N \subset S^\infty(C)$ be a complete Hilbert submanifold with $\dim T_mN^1 \le p < \infty$ for all $m \in N$.*

1. *If $\tau_h^q(N) \le 0$ for some $q \ge 2$ then N is a great sphere.*

2. *If* $\lambda_h(N) > 0$ *and* $\tau_h^q(N) \le \lambda_h^2(N)$ *for some* $q \ge 2$ *then* N *is a small sphere.*

Proof of Theorem 11: In case 1, by Theorem 10 we obtain $K_h(m) \le 0$, and the statement follows from Theorem 3 in [14].

The proof in case 2 is similar to proof of Theorem 5 for case (b). Assume the contrary: N is not a small sphere. By Theorem 1, there is a continuous normal vector field ξ of the length $\lambda_h(N)$ and strongly conformal nullity subspaces $T_c(\xi)$ of codimension $0 < \operatorname{codim} T_c(\xi) \le 2(p-1)(q-1)$. The leaves $\{L\}$ of the strongly conformal nullity distribution are small spheres of finite codimension in $S^\infty(C)$ of normal curvature $\lambda_h(N)$, see Proposition 5, and they are totally geodesic in N. Let L_1, L_2 be two sufficiently close leaves. Let $m_i \in L_i$ be the nearest points of these small spheres and $\gamma(t)$ $(0 \le t \le 1)$ be a shortest geodesic (an arc of a great circle) with the length $l = \operatorname{dist}(L_1, L_2)$ between the points $m_i \in L_i$. This geodesic is orthogonal to L_1 and L_2. Since the normals $\xi_1 = \dot\gamma\,(0), \xi_2 = \dot\gamma\,(1)$ to L_1, L_2 are tangent to N and orthogonal to mean curvature vectors of leaves, then $h_i(x_i, x_i) \perp \xi_i$, where $x_i \in T_{m_i} L_i$. Since the codimension of leaves in finite and dimension is infinite, there is unit vector $y_1 \in T_{m_1} L$ such that its parallel translation $y_1(t)$ along γ belongs to $T_{m_2} L$. From non-negativeness of the $2^{\rm nd}$ variation of the energy of γ we conclude $1/2E''(y_1) = -\int_0^1 K_{S^\infty|N}(\dot\gamma, y_1(t))dt \ge 0$, a contradiction to positiveness of curvature of a Hilbert sphere. ∎

Corollary 5: *A complete constant isotropic Hilbert submanifold* $N \subset S^\infty(C)$ *with dimension* $\dim T_m N^1 < \infty$ *is a small sphere.*

An application of Theorem 10 are the following statements. Their proofs are similar to proofs of Theorems 2–3.

Theorem 12: *Let* $N = N_1 \times N_2$ *be a product of Hilbert manifolds. Suppose that there exists* $m = (m_1, m_2) \in N$ *such that* $K_{N_i}(\sigma) \ge 0$ $(\sigma \subset T_{m_i} N_i)$ *and* $\tau^q_{N_i}(m_i) \le 1$ $(i = 1, 2)$ *for some* q*. Then, there is no isometric immersion of* N *into a Hilbert sphere* $S^\infty(C)$ *with* $C = 1 + Heav(q-4)$ *and* $\dim T_m N^1 < \infty$.

Case $q = 2$ of Theorem 12 can be proven without the assumption $K_{N_i}(\sigma) \ge 0$ $(\sigma \subset T_{m_i} N_i)$.

Theorem 13: *Let* $N = N_1 \times N_2$ *be a product of Hilbert manifolds. Suppose that there exists* $m = (m_1, m_2) \in N$ *such that* $K_{N_i}(m_i) \le C + \lambda_0^2$ $(i = 1, 2)$ *for some* $\lambda_0 > 0$ *and* $C > -\lambda_0^2$*. Then, there is no isometric immersion of* N *into a Hilbert space* $M^\infty(C)$ *of constant sectional curvature* C *with* $\lambda_h(m) \ge \lambda_0$ *and* $\dim T_m N^1 < \infty$.

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